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A strong structural stability of C_{2k+1} -free graphs. (English. English summary)

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All graphs considered are simple and undirected. For a graph F , the Turán number $\text{ex}(n, F)$ is the maximum number of edges in an F -free graph on n vertices. Z. Füredi and D. S. Gunderson [Combin. Probab. Comput. **24** (2015), no. 4, 641–645; MR3350026] showed that $\text{ex}(n, C_{2k+1}) = \lfloor n^2/4 \rfloor$ for $n \geq 4k - 2$, and that $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is the unique extremal graph for $n \geq 4k$. It is natural to ask how far a C_{2k+1} -free graph of nearly extremal size must be from being bipartite. Let $d_2(G)$ (resp., $\gamma_2(G)$) be the least number of vertices (resp., edges) whose deletion from G leaves a bipartite graph. Let $T^*(r, n)$ be the graph obtained from $K_{\lfloor (n-r+1)/2 \rfloor, \lceil (n-r+1)/2 \rceil}$ and K_r by identifying a vertex of one with a vertex of the other, so that $e(T^*(r, n)) = \lfloor (n-r+1)^2/4 \rfloor + \binom{r}{2}$. S. Ren et al. [SIAM J. Discrete Math. **38** (2024), no. 2, 1733–1756; MR4755075] showed that if $3 \leq r \leq 2k$ and $n \geq 318(r-2)^2k$, then every C_{2k+1} -free graph G on n vertices with $e(G) \geq e(T^*(r, n))$ satisfies $d_2(G) \leq r - 2$ and $\gamma_2(G) \leq \binom{\lceil r/2 \rceil}{2} + \binom{\lfloor r/2 \rfloor}{2}$, with equality in either bound only for $G \cong T^*(r, n)$.

In this paper, the authors obtain both bounds as a consequence of a more general structural result. Say that a graph G is a *suspension* to a graph B if the two have at most one vertex in common, called the *suspension point*, and no edge joins $V(G) \setminus V(B)$ to $V(B) \setminus V(G)$ (for example, when K_r is a suspension to $K_{\lfloor (n-r+1)/2 \rfloor, \lceil (n-r+1)/2 \rceil}$, the resulting graph is $T^*(r, n)$). The authors show that if $r \geq 3$, $2k \geq r + 4$ and $n \geq 20(r+2)^2k$, then any C_{2k+1} -free graph G on n vertices with $e(G) \geq e(T^*(r, n))$ is obtained from a bipartite graph $B = G[V_1, V_2]$ by attaching suspensions one at a time, in such a way that at most $r - 1$ vertices of G lie outside B , $e(B) \geq \lfloor (n-r+1)^2/4 \rfloor$, and

$$||V_i| - n/2| \leq \frac{\sqrt{(r-1)(n-r+1)}}{2}$$

for $i = 1, 2$. Moreover, exactly $r - 1$ vertices lie outside B precisely when $G \cong T^*(r, n)$. The authors prove this result by using the notion of a *strong $2k$ -core*, which is a subgraph in which any two vertices are joined both by an odd path of length at most $2k - 1$ and by an even path of length at most $2k - 2$; such subgraphs are easily enlarged, but under the above hypotheses they have at most r vertices, and the theorem is proved by induction on r .

Deleting all but one of the vertices outside B leaves a bipartite graph, as does deleting the edges lying inside the parts of a bipartition of each suspension, and these operations give the bounds on $d_2(G)$ and $\gamma_2(G)$ from [S. Ren et al., op. cit.] in the range $3 \leq r \leq 2k - 4$ and $n \geq 20(r+2)^2k$. The hypothesis on $e(G)$ amounts to

$$\text{ex}(n, C_{2k+1}) - e(G) \leq \frac{(r-1)(2n-r+1)}{4} - \binom{r}{2},$$

so what is described is the structure of C_{2k+1} -free graphs of edge deficiency $O(nk)$, rather than merely their distance from being bipartite. *Brahadeesh Sankarnarayanan*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.